Resit Exam — Functional Analysis (WIFA–08)

Tuesday 26 June 2018, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (7 + 5 + 10 + 3 = 25 points)

Define the following linear space:

$$X = \big\{x = (x_1, x_2, x_3, \dots) \,:\, x_k \in \mathbb{K},\ \|x\|_X < \infty\big\}, \quad \|x\|_X = |x_1| + \sum_{k=1}^{\infty} |x_{k+1} - x_k|.$$

(a) Prove that $\|\cdot\|_X$ is a norm on X.

(b) Recall the following Banach space from the lecture notes:

$$\ell^1 = \big\{x = (x_1, x_2, x_3, \dots) \,:\, x_k \in \mathbb{K},\ \|x\|_1 < \infty\big\}, \quad \|x\|_1 = \sum_{k=1}^{\infty} |x_k|.$$

Consider the linear map:

$$T : X \to \ell^1, \quad (x_1, x_2, x_3, \dots) \mapsto (x_1, x_2 - x_1, x_3 - x_2, \dots).$$

Show that T is bijective and $\|Tx\|_1 = \|x\|_X$ for all $x \in X$.

(c) Prove that $(X, \|\cdot\|_X)$ is a Banach space using that $(\ell^1, \|\cdot\|_1)$ is a Banach space.

(d) Show that the norms $\|\cdot\|_X$ and $\|\cdot\|_1$ are *not* equivalent on the space ℓ^1.

Problem 2 (6 + 4 + 4 + 4 + 7 = 25 points)

Consider the space $X = \mathcal{C}([0,1], \mathbb{K})$ with norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ and the following linear operator:

$$T : X \to X, \quad Tf(x) = \int_0^1 e^{x-t} f(t)\, dt.$$

(a) Show that T is compact.

(b) Show that $0 \in \sigma(T)$.

(c) Assume $\lambda \neq 0$. Show that if $Tf - \lambda f = g$, then $f = \alpha e^x - g/\lambda$ for some $\alpha \in \mathbb{K}$.

(d) Compute $(T - \lambda)^{-1} g$ by computing the constant α in terms of g and λ.

(e) Determine $\rho(T)$ and hence $\sigma(T)$.

Problem 3 (5 + 3 + 7 + 5 = 20 points)

(a) Formulate Baire's theorem for metric spaces.

Let $\|\cdot\|$ be any norm on the space of finitely supported sequences:

$$\mathcal{S} = \{x = (x_1, x_2, x_3, \dots) : x_k \in \mathbb{K}, \text{ there exists } N_x \in \mathbb{N} \text{ s.t. } x_k = 0 \text{ for } k \geq N_x\}.$$

Prove the following statements:

(b) $\mathcal{S}_n = \{x \in \mathcal{S} : x_k = 0 \text{ for all } k \geq n\}$ is closed for each $n \in \mathbb{N}$;

(c) $\mathcal{S}_n$ is nowhere dense for each $n \in \mathbb{N}$;

(d) $\mathcal{S}$ is *not* a Banach space.

Problem 4 (4 + 6 = 10 points)

Let X be a Hilbert space and let $V \subset X$ be a subset.

(a) For $v \in V$ define the linear map $f_v : X \to \mathbb{K}$ by $f_v(x) = (x, v)$. Show that

$$\|f_v\| = \|v\|.$$

(b) Assume that for each $x \in X$ there exists a constant $M_x \geq 0$ such that

$$|(v, x)| \leq M_x \quad \text{for all } v \in V.$$

Use the uniform boundedness principle to prove that the set V is bounded.

Problem 5 (4 + 6 = 10 points)

Let X be a normed linear space and let $x \in X$. Define the map

$$F_x : X' \to \mathbb{K}, \quad F_x(f) = f(x), \quad f \in X',$$

and define the map $J : X \to X''$ by $J(x) = F_x$.

(a) Prove that $F_x : X' \to \mathbb{K}$ is linear and that $\|F_x\| = \|x\|$.

(b) Assume that X is *not* a Banach space. Explain how the map J can be used to construct a completion of X.

End of test (90 points)

Solution of problem 1 (7 + 5 + 10 + 3 = 25 points)

(a) It is clear that $\|x\|_X \geq 0$ for any $x \in X$. If $\|x\|_X = 0$, then

$$|x_1| = 0 \quad \text{and} \quad |x_{k+1} - x_k| = 0 \quad \text{for all } k \in \mathbb{N},$$

which implies that $x_k = 0$ for all $k \in \mathbb{N}$ so that $x = 0$.
(2 points)

For $\lambda \in \mathbb{K}$ and $x \in X$ we have $\lambda x = (\lambda x_1, \lambda x_2, \dots)$ so that

$$\|\lambda x\|_X = |\lambda x_1| + \sum_{k=1}^{\infty} |\lambda(x_{k+1} - x_k)| = |\lambda||x_1| + |\lambda| \sum_{k=1}^{\infty} |x_{k+1} - x_k| = |\lambda| \|x\|_X.$$

(2 points)

For $x, y \in X$ we have $x + y = (x_1 + y_1, x_2 + y_2, \dots)$ so that

$$\begin{aligned} \|x + y\|_X &= |x_1 + y_1| + \sum_{k=1}^{\infty} |(x_{k+1} - x_k) + (y_{k+1} - y_k)| \\ &\leq |x_1| + |y_1| + \sum_{k=1}^{\infty} \big(|x_{k+1} - x_k| + |y_{k+1} - y_k|\big) \\ &= |x_1| + |y_1| + \sum_{k=1}^{\infty} |x_{k+1} - x_k| + \sum_{k=1}^{\infty} |y_{k+1} - y_k| \\ &= \|x\|_X + \|y\|_X. \end{aligned}$$

(3 points)

(b) If $Tx = 0$, then $x_1 = 0$ and $x_{k+1} - x_k = 0$ for all $k \in \mathbb{N}$, which implies that $x_k = 0$ for all $k \in \mathbb{N}$ so that $x = 0$. This shows that T is injective.
(2 points)

Let $y = (y_1, y_2, \dots) \in \ell^1$ and set $x = (x_1, x_2, \dots)$ by $x_k = y_1 + \cdots + y_k$, then $Tx = y$ and

$$\|x\|_X = |x_1| + \sum_{k=1}^{\infty} |x_{k+1} - x_k| = |y_1| + \sum_{k=1}^{\infty} |y_{k+1}| = \|y\|_1 < \infty,$$

which shows that $x \in X$. Therefore, T is surjective. The equality $\|Tx\|_1 = \|x\|_X$ for all $x \in X$ is trivial.
(3 points)

(c) Let x^n be a Cauchy sequence in $(X, \|\cdot\|_X)$ and set $y^n = Tx^n$. Let $\epsilon > 0$ be arbitrary, then there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \quad \Rightarrow \quad \|y^n - y^m\|_1 = \|T(x^n - x^m)\|_1 = \|x^n - x^m\|_X \leq \epsilon,$$

which shows that y^n is a Cauchy sequence in $(\ell^1, \|\cdot\|_1)$.
(4 points)

Since $(\ell^1, \|\cdot\|_1)$ is complete there exists $y \in \ell^1$ such that $\|y^n - y\|_1 \to 0$ as $n \to \infty$. Now let $x = T^{-1}y$, then since T^{-1} is also isometric we have

$$\|x^n - x\|_X = \|T^{-1}(y^n - y)\|_X = \|y^n - y\|_1 \to 0.$$

(6 points)

(d) Consider the sequence $x^n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots)$. On the one hand we have

$$\|x^n\|_1 = \sum_{k=1}^{n} \frac{1}{k} \to \infty \quad \text{as } n \to \infty.$$

On the other hand we have

$$\|x^n\|_X = 1 + \left|\frac{1}{2} - 1\right| + \left|\frac{1}{3} - \frac{1}{2}\right| + \cdots + \left|\frac{1}{n} - \frac{1}{n-1}\right| + \left|0 - \frac{1}{n}\right| = 2.$$

Therefore, there is no constant $C > 0$ such that $\|x\|_1 \le C\|x\|_X$ for all $x \in \ell^1$.
(3 points)

Solution of problem 2 (6 + 4 + 4 + 4 + 7 = 25 points)

(a) *Solution 1.* For all $x \in [0,1]$ we have that

$$|Tf(x)| = \left| \int_0^1 e^{x-t} f(t)\,dt \right| \le \int_0^1 e^{x-t} |f(t)|\,dt \le \|f\|_\infty \int_0^1 e^{x-t}\,dt = \|f\|_\infty e^x (1-e^{-1}).$$

Taking the supremum over all $x \in [0,1]$ gives

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |Tf(x)| \le (e-1)\|f\|_\infty,$$

which shows that T is a bounded operator.
(3 points)

Also note that for any $f \in X$ we have $Tf \in \operatorname{span}\{e^x\}$, which shows that $\dim \operatorname{ran} T = 1$. Together with the boundedness of T this implies that T is a compact operator.
(3 points)

Solution 2. Recall the following theorem from the lecture notes: if $K : [a,b] \times [a,b] \to \mathbb{K}$ is a continuous function, then the Fredholm operator

$$T : X \to X, \quad Tf(x) = \int_a^b K(x,t) f(t)\,dt$$

is a compact operator. Clearly, the function $K(x,t) = e^{x-t}$ satisfies the hypothesis of this theorem.
(6 points)

(b) *Solution 1.* Since T is compact and X is infinite-dimensional a theorem of the lecture notes guarantees that $0 \in \sigma(T)$.
(4 points)

Solution 2. Since $\dim \operatorname{ran} T = 1$ and X is infinite-dimensional we have that $\operatorname{ran} T$ is not dense in X. This means that $0 \notin \rho(T)$, or, equivalently, $0 \in \sigma(T)$.
(4 points)

Solution 3. Let $f \in X$ be nontrivial and satisfy $\int_0^1 f(t)\,dt = 0$. For example, let $f(x) = x - \frac{1}{2}$. Then the function $g(x) = e^x f(x)$ belongs to $\ker T$. This implies that $0 \in \sigma_p(T) \subset \sigma(T)$.
(4 points)

(c) Note that for any f we have that $Tf = \beta e^x$ where $\beta = \int_0^1 e^{-t} f(t)\,dt$ is a constant depending on f. If $Tf - \lambda f = g$, then $f = Tf/\lambda - g/\lambda$ and f is necessarily of the form $f = \alpha e^x - g/\lambda$, where $\alpha = \beta/\lambda$.
(4 points)

(d) Computing $f = (T-\lambda)^{-1} g$ means finding $f \in X$ such that $Tf - \lambda f = g$. Part (c) implies that there exists a constant $\alpha \in \mathbb{K}$ such that $f(x) = \alpha e^x - g(x)/\lambda$. In this case the equation $Tf - \lambda f = g$ reads as

$$\int_0^1 e^{x-t} \left(\alpha e^t - \frac{g(t)}{\lambda} \right) dt - \lambda \alpha e^x + g(x) = g(x),$$

or, equivalently,

$$\alpha = -\frac{1}{\lambda(\lambda-1)} \int_0^1 e^{-t} g(t)\, dt.$$

This gives

$$(T-\lambda)^{-1} g = -\frac{1}{\lambda} g(x) - \frac{1}{\lambda(\lambda-1)} \int_0^1 e^{x-t} g(t)\, dt.$$

(4 points)

(e) Note that for $\lambda \notin \{0,1\}$ the operator

$$S_\lambda = -\frac{1}{\lambda} - \frac{1}{\lambda(\lambda-1)} T$$

is well-defined and bounded since it is a linear combination of two bounded operators (namely the identity and T). A straightforward computation shows that

$$(T-\lambda) S_\lambda = S_\lambda (T-\lambda) = I,$$

which means that $(T-\lambda)^{-1} = S_\lambda \in B(X)$ for all $\lambda \notin \{0,1\}$. This implies $\mathbb{K} \setminus \{0,1\} \subset \rho(T)$.
(5 points)

Since $\{0,1\} \subset \sigma(T)$ we have in fact that $\rho(T) = \mathbb{K} \setminus \{0,1\}$ and $\sigma(T) = \{0,1\}$.
(2 points)

Solution of problem 3 (5 + 3 + 7 + 5 = 20 points)

(a) *Alternative 1.* Let X be a complete metric space and let $O \subset X$ be nonempty and open. Then O is nonmeager.
(5 points)

Alternative 2. A complete metric space cannot be written as the countable union of nowhere dense subsets.
(5 points)

(b) Note that $\mathcal{S}_n = \{x \in \mathcal{S} : x_k = 0 \text{ for all } k \geq n\}$ is a finite-dimensional subspace of the normed linear space $\mathcal{S}$. This implies that $\mathcal{S}_n$ is closed.
(3 points)

(c) We need to prove that $\operatorname{int} \overline{\mathcal{S}_n} = \emptyset$, or, equivalently, since $\mathcal{S}_n$ is closed, that $\operatorname{int} \mathcal{S}_n = \emptyset$.
(2 points)

If $x \in \operatorname{int} \mathcal{S}_n$ then there exists $\varepsilon > 0$ such that

$$\{y \in \mathcal{S} : \|y - x\| < \varepsilon\} \subset \mathcal{S}_n.$$

Let $z \in \mathcal{S}$ be nonzero and define $\widetilde{z} = x + \frac{1}{2}\varepsilon z/\|z\|$ then

$$\|\widetilde{z} - x\| = \tfrac{1}{2}\varepsilon,$$

which implies that $\widetilde{z} \in \mathcal{S}_n$. In turn, this implies that

$$z = \frac{2\|z\|}{\varepsilon}(\widetilde{z} - x) \in \mathcal{S}_n$$

so that $\mathcal{S} = \mathcal{S}_n$, which is a contradiction. Hence, $\operatorname{int} \mathcal{S}_n = \emptyset$.
(5 points)

(d) If $\mathcal{S}$ is a Banach space, then it is also a complete metric space. Since

$$\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$$

it would follow from Baire's theorem that at least one of the sets $\mathcal{S}_n$ is *not* nowhere dense. This contradicts the conclusion of part (c). Hence, we conclude that $\mathcal{S}$ is *not* a Banach space.
(5 points)

Solution of problem 4 (4 + 6 = 10 points)

(a) For $x \in X$ the Cauchy-Schwarz inequality gives $|f_v(x)| = |(x, v)| \leq \|x\| \|v\|$, which implies that

$$\sup_{x \neq 0} \frac{|f_v(x)|}{\|x\|} \leq \|v\|.$$

(3 points)

For $x = v$ we have

$$\frac{|f_v(x)|}{\|x\|} = \frac{|(v, v)|}{\|v\|} = \|v\|.$$

Hence, $\|f_v\| = \|v\|$.
(1 point)

(b) For any $x \in X$ there exists a constant $M_x \geq 0$ such that

$$|f_v(x)| = |(x, v)| = |(v, x)| \leq M_x,$$

which implies that

$$\sup_{v \in V} |f_v(x)| < \infty \quad \text{for all } x \in X.$$

(3 points)

By part (a) and the uniform boundedness principle we have

$$\sup_{v \in V} \|v\| = \sup_{v \in V} \|f_v\| < \infty,$$

which implies that the set V is bounded.
(3 points)

Problem 5 (4 + 6 = 10 points)

(a) For $f, g \in X'$ and $\lambda, \mu \in \mathbb{K}$ we have

$$F_x(\lambda f + \mu g) = (\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x) = \lambda F_x(f) + \mu F_x(g),$$

which shows that $F_x : X' \to \mathbb{K}$ is a linear map.
(2 points)

We have

$$\|F_x\| = \sup_{f \in X', f \neq 0} \frac{|F_x(f)|}{\|f\|} = \sup_{f \in X', f \neq 0} \frac{|f(x)|}{\|f\|} = \|x\|,$$

where the last equality is a consequence of the Hahn-Banach theorem.
(2 points)

(b) The operator $J : X \to X''$ is an isometry and hence injective. This means that $J(X)$ is a copy of X inside X''. Set $\widetilde{X} = \overline{J(X)}$. Since X'' is a Banach space and $\widetilde{X}$ is closed in X'' it follows that $\widetilde{X}$ is a Banach space. If x_n is a Cauchy sequence in X, then Jx_n is a Cauchy sequence in $\widetilde{X}$ (since J is isometric) and hence convergent. In this way, every Cauchy sequence in X has a limit in the larger space $\widetilde{X}$ and hence the latter space can be considered as a completion of X.
(6 points)